- A system with input x and output y is BIBO stable if, for every bounded x, y is bounded (i.e., $|x(n)| < \infty$ for all n implies that $|y(n)| < \infty$ for all n.(
- To show that a system is *BIBO stable*, we must show that *every bounded input* leads to a *bounded output*.
- To show that a system is *not BIBO stable*, we need only find a single *bounded input* that leads to an *unbounded output*.
- In practical terms, a BIBO stable system is *well behaved* in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have *serious safety issues*. For example, an iPod with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized Apple customer and one big lawsuit.

- A system H is said to be time invariant (TI) if, for every sequence x and every integer n_0, the following condition holds:

$$y(n-n_0) = Hx'(n) \quad \text{where} \quad y = Hx \quad \text{and} \quad x'(n) = x(n-n_0)$$

  (i.e., H *commutes with time shifts*).
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an *identical time shift* in the output.
- A system that is not time invariant is said to be time varying.
- In simple terms, a time invariant system is a system whose behavior *does not change* with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much *easier to design and analyze*, since their behavior does not change with respect to time.

- A system H is said to be additive if, for all sequences x_1 and x_2, the following condition holds:

$$H(x_1 + x_2) = Hx_1 + Hx_2$$

)i.e., H *commutes with sums*.(

- A system H is said to be homogeneous if, for every sequence x and every complex constant a, the following condition holds:

$$H(ax) = aHx$$

)i.e., H *commutes with multiplication by a constant*.(

- A system that is both additive and homogeneous is said to be linear.
- In other words, a system H is *linear*, if for all sequences x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$H(a_1x_1 + a_2x_2) = a_1Hx_1 + a_2Hx_2$$

)i.e., H *commutes with linear combinations*.(

- The linearity property is also referred to as the superposition property.
- Practically speaking, linear systems are much *easier to design and analyze* than nonlinear systems.

Part 8

Discrete-Time Linear Time-Invariant (LTI) Systems

- In engineering, linear-time invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI. In
- practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 8.1

Convolution

- The (DT) convolution of the sequences x and h, denoted $x*h$, is defined as the sequence

$$x*h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

- The convolution $x*h$ evaluated at the point n is simply a weighted sum of elements of x, where the weighting is given by h time reversed and shifted by n.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in the theory of (DT) systems.
- In particular, convolution has a special significance in the context of (DT) LTI systems.

- To compute the convolution

$$x*h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

we proceed as follows:

1. • Plot $x(k)$ and $h(n-k)$ as a function of k.
2. • Initially, consider an arbitrarily large negative value for n. This will result in •$h(n-k)$ being shifted very far to the left on the time axis. Write the mathematical expression for $x*h(n$.(
3. •Increase n gradually until the expression for $x*h(n)$ changes form.
4. Record the interval over which the expression for $x*h(n)$ was valid.
5. •Repeat steps 3 and 4 until n is an arbitrarily large positive value. This corresponds to $h(n-k)$ being shifted very far to the right on the time axis.
6. •The results for the various intervals can be combined in order to obtain an expression for $x*h(n)$ for all n.

- The convolution operation is *commutative*. That is, for any two sequences x and h,

$$x * h = h * x.$$

- The convolution operation is *associative*. That is, for any sequences x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is *distributive* with respect to addition. That is, for any sequences x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

- For any sequence x,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = x * \delta(n).$$

- Thus, any sequence x can be written in terms of an expression involving δ.
- Moreover, δ is the *convolutional identity*. That is, for any sequence x,

$$x * \delta = x.$$

- The convolution of two periodic sequences is usually not well defined. This
- motivates an alternative notion of convolution for periodic sequences known as circular convolution.
- The circular convolution (also known as the DT periodic convolution) of the T-periodic sequences x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(n) = \sum_{k=\langle N \rangle} x(k)h(n-k) = \sum_{k=0}^{N-1} x(k)h(\operatorname{mod}(n-k,N)),$$

where $\operatorname{mod}(a,b)$ is the remainder after division when a is divided by b.

- The circular convolution and (linear) convolution of the N-periodic sequences x and h are related as follows:

$$x \circledast h(n) = x_0 * h(n) \quad \text{where} \quad x(n) = \sum_{k=-\infty}^{\infty} x_0(n-kN)$$

(i.e., $x_0(n)$ equals $x(n)$ over a single period of x and is zero elsewhere).

Section 8.2

Convolution and LTI Systems

- The response h of a system H to the input δ is called the impulse response of the system (i.e., $h = H\{\delta\}$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply *computes a convolution*.
- Furthermore, a LTI system is *completely characterized* by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.

- The response s of a system $\mathcal{H}$ to the input u is called the step response of the system (i.e., $s = \mathcal{H}\{u\}$).
- The impulse response h and step response s of a system are related as

$$h(n) = s(n) - s(n-1).$$

- Therefore, the impulse response of a system can be determined from its step response by (first-order) differencing.

- Often, it is convenient to represent a (DT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$